THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 6

§42) 2) d) Since $\lim_{t\to\infty} |e^{-zt}| = \lim_{t\to\infty} e^{-(\operatorname{Re} z)t} = 0$ for $\operatorname{Re} z > 0$, we have $\lim_{t\to\infty} e^{-zt} = 0$. Therefore,

$$\int_0^\infty e^{-zt}dt = \frac{-1}{z}\int_0^\infty e^{-zt}d(-zt) = \left[\frac{-1}{z}e^{-zt}\right]_0^\infty = \frac{1}{z}.$$

§42) 4) Note that

$$\int_0^\pi e^{(1+i)x}dx = \int_0^\pi e^x \cos x dx + i\int_0^\pi e^x \sin x dx. \tag{1}$$

Since

$$\int_0^\pi e^{(1+i)x}dx = \left[\frac{1}{1+i}e^{(1+i)x}\right]_0^\pi = \frac{e^{\pi+i\pi}-1}{1+i} = -\frac{e^\pi+1}{1+i} = -\frac{(e^\pi+1)}{2} + i\frac{(e^\pi+1)}{2},$$

by comparing the real part and imaginary part on both sides of (1), we have

$$\int_0^\pi e^x \cos x dx = -\frac{(e^\pi+1)}{2} \text{ and } \int_0^\pi e^x \sin x dx = \frac{(e^\pi+1)}{2}.$$

§43) 4) The equation of straight line in τt plane passing through the points (α, a) and (β, b) is given by

$$\frac{t-a}{\tau-\alpha} = \frac{b-a}{\beta-\alpha}$$

$$\Longrightarrow \qquad t = \frac{b-a}{\beta-\alpha}\tau + \frac{a\beta-b\alpha}{\beta-\alpha}$$

In particular, we may take $\phi(\tau)$ to be

$$\phi(\tau) = \frac{b-a}{\beta-\alpha}\tau + \frac{a\beta-b\alpha}{\beta-\alpha}.$$

Clearly it is bijective and strictly increasing on $[\alpha, \beta]$.

§43) 5) Write $w(x,y) = u(x,y) + iv(x,y)$ and $z(t) = x(t) + iy(t)$.

If $w(t) = f[z(t)] = u(x(t), y(t)) + iv(x(t), y(t))$, then by Chain rule, we have

$$\begin{aligned} w'(t) &= \frac{d}{dt}u(x(t),y(t)) + i\frac{d}{dt}v(x(t),y(t)) \\ &= [u_x(z(t))x'(t) + u_y(z(t))y'(t)] + i[v_x(z(t))x'(t) + v_y(z(t))y'(t)] \end{aligned}$$

By Cauchy-Riemann equation, we have

$$\begin{aligned} w'(t) &= [u_x(z(t))x'(t) + u_y(z(t))y'(t)] + i[v_x(z(t))x'(t) + v_y(z(t))y'(t)] \\ &= [u_x(z(t))x'(t) - v_x(z(t))y'(t)] + i[v_x(z(t))x'(t) + u_x(z(t))y'(t)] \\ &= [u_x(z(t)) + iv_x(z(t))][x'(t) + iy'(t)] \\ &= f'[z(t)]z'(t). \end{aligned}$$

§46) 1) For the function $f(z) = \dfrac{z+2}{z}$,

a) $\displaystyle\int_C f(z)dz = \int_0^\pi \frac{2e^{i\theta}+2}{2e^{i\theta}} d(2e^{i\theta}) = 2\int_0^\pi (e^{i\theta}+1)di\theta = 2\left[e^{i\theta}+i\theta\right]_0^\pi = -4+2\pi i.$

b) $\displaystyle\int_C f(z)dz = \int_\pi^{2\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} d(2e^{i\theta}) = 2\int_\pi^{2\pi} (e^{i\theta}+1)di\theta = 2\left[e^{i\theta}+i\theta\right]_\pi^{2\pi} = 4+2\pi i.$

c) By a) and b), we have $\displaystyle\int_C f(z)dz = (-4+2\pi i) + (4+2\pi i) = 4\pi i.$

§46) 4) Parametrize the curve C by $\gamma(t) = t + t^3 i$, where $t \in [-1,1]$. We have

$$\begin{aligned}\int_C f(z)dz &= \int_{-1}^0 f(\gamma(t))\gamma'(t)dt + \int_0^1 f(\gamma(t))\gamma'(t)dt \\ &= \int_{-1}^0 (1+3t^2 i)dt + \int_0^1 (4t^3)(1+3t^2 i)dt \\ &= \left[t+t^3 i\right]_{-1}^0 + \left[t^4+2t^6 i\right]_0^1 \\ &= -(-1-i) + (1+2i) \\ &= 2+3i.\end{aligned}$$

§46) 9) a) For the principal branch of $z^{-3/4}$,

$$\begin{aligned}\int_C f(z)dz &= \int_{-\pi}^{\pi} \exp\left[-\frac{3}{4}\operatorname{Log}(e^{i\theta})\right] ie^{i\theta}d\theta \\ &= \int_{-\pi}^{\pi} e^{-\frac{3}{4}i\theta}(ie^{i\theta})d\theta \\ &= \int_{-\pi}^{\pi} e^{i\frac{\theta}{4}} i d\theta \\ &= \left[4e^{i\frac{\theta}{4}}\right]_{-\pi}^{\pi} \\ &= 4\sqrt{2}i.\end{aligned}$$

b) For the branch $\arg z \in (0, 2\pi)$ of $z^{-3/4}$, similarly,

$$\begin{aligned}\int_C f(z)dz &= \int_0^{2\pi} \exp\left[-\frac{3}{4}\operatorname{Log}(e^{i\theta})\right] ie^{i\theta}d\theta \\ &= \left[4e^{i\frac{\theta}{4}}\right]_0^{2\pi} \\ &= -4+4i.\end{aligned}$$

§46) 13) For $n \in \mathbb{Z}$,

$$\begin{aligned}
\int_{C_0} (z - z_0)^{n-1} dz &= \int_0^{2\pi} R^{n-1} e^{i(n-1)\theta} (Re^{i\theta}) i d\theta \\
&= \begin{cases} \int_0^{2\pi} i d\theta & \text{if } n = 0 \\ R^n \int_0^{2\pi} e^{in\theta} i d\theta & \text{if } n \neq 0; \end{cases} \\
&= \begin{cases} 2\pi i & \text{if } n = 0 \\ R^n \left[\dfrac{e^{in\theta}}{n} \right]_0^{2\pi} & \text{if } n \neq 0; \end{cases} \\
&= \begin{cases} 2\pi i & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}
\end{aligned}$$